THE CHINESE UNIVERSITY OF HONG KONG
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MATH2050C Mathematical Analysis I

Tutorial 8 (March 25)

Definition. Let $A \subseteq \mathbb{R}$. A point $c \in \mathbb{R}$ is said to be a **cluster point** of A if given any $\delta > 0$, there exists $x \in A$, $x \neq c$ such that $|x - c| < \delta$.

Remarks. (1) Equivalently, c is a cluster point of A if and only if

$$V_\delta(c) \cap A \setminus \{c\} \neq \emptyset \quad \text{for any } \delta > 0.$$

(2) A cluster point of A may or may not be an element of A.

Example 1. The following are some subsets of $\mathbb{R}$ and their sets of cluster points.

(a) The set of cluster points of $\mathbb{N}$ is $\emptyset$

(b) The set of cluster points of $(1, 2) \cap \mathbb{Q}$ is $[1, 2]$.

Definition. Let $A \subseteq \mathbb{R}$, and let c be a cluster point of A. For a function $f : A \to \mathbb{R}$, a real number L is said to be a **limit of f at c** if, given any $\varepsilon > 0$, there exists a $\delta > 0$ such that if $x \in A$ and $0 < |x - c| < \delta$, then $|f(x) - L| < \varepsilon$.

In this case, we write

$$\lim_{x\to c} f = L, \quad \lim_{x\to c} f(x) = L \quad \text{or} \quad f(x) \to L \text{ as } x \to c.$$

Theorem. *If $f\colon A \to \mathbb{R}$ and if c is a cluster point of A, then f can have only one limit at c.*

Example 2. Use the ε-δ definition of limit to show that $\lim_{x\to -3}(x^2 + 4x) = -3$.

Solution. Note that $|(x^2 + 4x) - (-3)| = |x^2 + 4x + 3| = |x + 1||x + 3|$.

If $|x + 3| < 1$, then $|x + 1| = |(x + 3) - 2| \leq |x + 3| + 2 < 3$.

Let $\varepsilon > 0$ be given. Take $\delta := \min\{\varepsilon/3, 1\}$. Now if $0 < |x - (-3)| < \delta$, then

$$|(x^2 + 4x) - (-3)| = |x + 1||x + 3| < 3 \cdot \frac{\varepsilon}{3} = \varepsilon.$$

Hence $\lim_{x\to -3}(x^2 + 4x) = -3$. ◀

Example 3. Use the ε-δ definition of limit to show that $\lim_{x\to 2} \dfrac{x + 6}{x^2 - 2} = 4$.

Solution. Clearly $f(x) := \dfrac{x + 6}{x^2 - 2}$ has a natural domain $\mathbb{R} \setminus \{\pm\sqrt{2}\}$, which has 2 as a cluster point.

For $x \in \mathbb{R} \setminus \{\pm\sqrt{2}\}$,

$$|f(x) - 4| = \left|\frac{x+6}{x^2-2} - 4\right| = \frac{|4x^2 - x - 14|}{|x^2-2|} = \frac{|4x+7|}{|x^2-2|} \cdot |x-2|.$$

If $|x-2| < \frac{1}{2}$, then

$$\frac{3}{2} < x < \frac{5}{2} \implies \frac{1}{4} < x^2 - 2 < \frac{17}{4},$$

and

$$|4x+7| = |4(x-2) + 15| \leq 4|x-2| + 15 \leq 20.$$

Let $\varepsilon > 0$ be given. Take $\delta := \min\left\{\frac{\varepsilon}{80}, \frac{1}{2}\right\}$. Now if $0 < |x-2| < \delta$, then

$$|f(x) - 4| = \frac{|4x+7|}{|x^2-2|} \cdot |x-2| < \frac{20}{1/4} \cdot \frac{\varepsilon}{80} = \varepsilon.$$

Hence $\lim_{x\to 2} \frac{x+6}{x^2-2} = 4$. ◀

Classwork

Use the ε-δ definition of limit to establish the following limits.

(a) $\lim_{x\to 0} \frac{x^2}{|x|} = 0$.

(b) $\lim_{x\to 2} \frac{3x-4}{x^2-3} = 2$.

Solution. (a) Let $\varepsilon > 0$. Take $\delta = \varepsilon$. If $0 < |x - 0| < \delta$, then $\left|\frac{x^2}{|x|} - 0\right| = |x| < \delta = \varepsilon$. So $\lim_{x\to 0} \frac{x^2}{|x|} = 0$.

(b) For $x \neq \pm\sqrt{3}$,

$$\left|\frac{3x-4}{x^2-3} - 2\right| = \frac{|2x+1|}{|x^2-3|}|x-2|.$$

If $|x-2| < 1/4$, then

$$\frac{7}{4} < x < \frac{9}{4} \implies \frac{1}{16} < x^2 - 3 < \frac{33}{16},$$

and

$$|2x+1| \leq |2(x-2) + 5| \leq 2|x-2| + 5 < 10.$$

Let $\varepsilon > 0$. Take $\delta = \min\{\varepsilon/160, 1/4\}$. Now if $0 < |x-2| < \delta$, then

$$\left|\frac{3x-4}{x^2-3} - 2\right| = \frac{|2x+1|}{|x^2-3|}|x-2| < \frac{10}{1/16} \cdot \frac{\varepsilon}{160} < \varepsilon.$$

Hence $\lim_{x\to 2} \frac{3x-4}{x^2-3} = 2$.

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